Game Theory

Lecture 2: Best Response, Nash Equilibrium, and Epsilon-Nash

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Course textbooks

- Bonanno, G. (2024). *Game Theory (3rd ed.)*. University of California, Davis. Received from: GT Book
- Axelrod, R. (1984). *The Evolution of Cooperation*. Basic Books. Received from: Axelrod Article
- Nisan, N., Roughgarden, T., Tardos, É., & Vazirani, V. V. (2007). *Algorithmic Game Theory*. Cambridge University Press. Received from: AGT Book
- Myerson, R. B. (1991). Game Theory: Analysis of Conflict. Harvard University Press. Received from: GT Book 2
- F. Christianos et al., *Multi-Agent Reinforcement Learning: Foundations and Modern Approaches*, 2023. Received from: MARL Book.pdf
- Shoham, Y., & Leyton-Brown, K. (2008). Multiagent Systems: Algorithmic, Game-Theoretic, and Logical Foundations. Cambridge University Press Received from: MARL Book.pdf
- `nashpy` documentation (readthedocs) Link: NashPy Docs

Previously on Lecture 1

- Defined normal-form games, best response, dominance, and iterated elimination.
- Explored classic 2x2 games and their strategic structure.
- Introduced repeated games and the role of history in strategies.
- Discussed the importance of best response correspondences for equilibrium concepts.

Lecture Overview

- Mixed Strategies and the Indifference Principle
- Fixed Point Theorems and Nash's Existence Result
- Epsilon-Nash Equilibria
- Computational tools (`NashPy`)
- Fixed point theorems: Banach, Brouwer, Kakutani, and their connection to Nash

Outline

Normal-Form Game and Payoffs

- Players: $i \in \{1, ..., N\}$
- Actions: A_i finite, strategy $\pi_i \in \Delta(A_i)$
- Payoff: $R_i(a_i, a_{-i})$ for pure, extended linearly to mixed
- Joint strategy $\pi = (\pi_1, ..., \pi_N)$

Iterated Games

- A single-shot normal-form game captures one simultaneous move
- But many situations are **repeated** over time (e.g. pricing, traffic, negotiations)
- Iterated game = repeated play of the same stage game
  - Payoffs may be summed, averaged, or discounted
  - Allows strategies that condition on history (e.g. "tit-for-tat")
- Repetition introduces new equilibria beyond the one-shot case (Folk Theorems)

Axelrod's Tournament (Iterated Prisoner's Dilemma)

- Organized by Robert Axelrod in the 1980s
- Participants submitted computer programs to play repeated Prisoner's Dilemma
- Famous strategies:
  - **Always Defect** (greedy)
  - **Always Cooperate** (naïve)
  - **Tit-for-Tat** (cooperate first, then copy opponent's last move)
- **Results:** Tit-for-Tat won, showing that cooperation can emerge among self-interested agents
- In iterated settings, history-dependent strategies matter

Tournament Scores in Axelrod's Tournament

Prog.	TFT	T&C	NY	GR	SH	S&R	FR	DA	GR	DO	FE	JO	TU	NA	RAN	Mean	Rank Point	No. of Wins	Rank Wins
TFT	600	595	600	600	600	595	600	600	597	597	280	225	279	359	441	**504**	**1**	**0**	**15**
T&C	600	596	600	601	600	596	600	600	310	601	271	213	291	455	573	**500**	**2**	**11**	**2**
NY	600	595	600	600	600	595	600	600	433	158	354	374	347	368	464	**486**	**3**	**1**	**13.5**
GR	600	595	600	600	600	594	600	600	376	309	289	236	305	426	507	**482**	**4**	**4**	**6**
SH	600	595	600	600	600	595	600	600	348	271	274	272	265	448	543	**481**	**5**	**3**	**11.5**
S&R	600	596	600	602	600	596	600	600	319	200	252	249	280	480	592	**478**	**6**	**10**	**3.5**
FR	600	595	600	600	600	595	600	600	307	207	235	213	263	489	598	**473**	**7**	**6**	**8**
DA	600	595	600	600	600	595	600	600	307	194	238	247	253	450	598	**472**	**8**	**4**	**9.5**
GR	597	305	462	375	348	314	302	302	588	625	268	238	274	466	548	**401**	**9**	**5**	**9.5**
DO	597	591	398	289	261	215	202	239	555	202	436	540	243	487	604	**391**	**10**	**6**	**6**
FE	285	271	426	286	297	255	235	239	274	704	246	236	272	420	467	**328**	**11**	**12**	**3.5**
JO	230	214	409	237	286	254	213	252	244	634	236	224	273	390	469	**304**	**12**	**10**	**1**
TU	284	287	415	293	318	271	243	229	278	193	271	260	273	426	478	**301**	**13**	**6**	**6**
NA	362	231	397	273	230	149	133	173	187	133	317	366	345	413	526	**282**	**14**	**2**	**11.5**
RAN	442	142	407	313	219	141	108	137	189	102	360	416	419	300	450	**276**	**15**	**1**	**13.5**

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Solutions and Solution Concepts

- **Solution**: Prediction of (equilibrium) outcome
- A **solution concept** = a rule to predict how rational players will play
- Desiderata:
  - Consistency across players
  - Robustness to deviations
  - Predictive power in real-world applications
- Examples:
  - **Dominant strategies**: strictly best regardless of others
  - **Best response**: optimal given beliefs about opponents
  - **Nash equilibrium**: mutual best responses
- Solution concepts differ in strength and applicability

Outline

Pure and Mixed Strategies

- **Pure strategy:** Choose a single action deterministically.
- **Mixed strategy:** Probability distribution over available actions.
- Mixed strategies expand the strategy space to convex sets (simplices).
- Many games (e.g. Matching Pennies) have **no pure NE**.
- Mixed strategies guarantee existence of equilibrium [Nash 1950].

Why might randomization be rational in some games?

Example: Pure vs Mixed Strategies

	H	T
H	(1,-1)	(-1,1)
T	(-1,1)	(1,-1)

- No pair of pure strategies is stable (one player always wants to deviate).
- **Mixed strategy solution:** Each player randomizes: H with 0.5, T with 0.5.

What happens if you try to "outguess" your opponent in this game?

Best Response: Definition

For player i and opponents' mixed strategy π_{-i},

$$BR_i(\pi_{-i}) = \arg \max_{\pi_i \in \Delta(A_i)} R_i(\pi_i, \pi_{-i}).$$

- May be multi-valued
- Always nonempty for finite games
- Contains all optimal mixtures against π_{-i}

Best Response in Pure Strategies

With opponents fixed at a pure action a_{-i},

$$BR_i(a_{-i}) = \arg\max_{a_i \in A_i} R_i(a_i, a_{-i}).$$

- Quick to read off in a payoff matrix
- Underline the highest entry in each row or column as appropriate

Best Response as a Correspondence

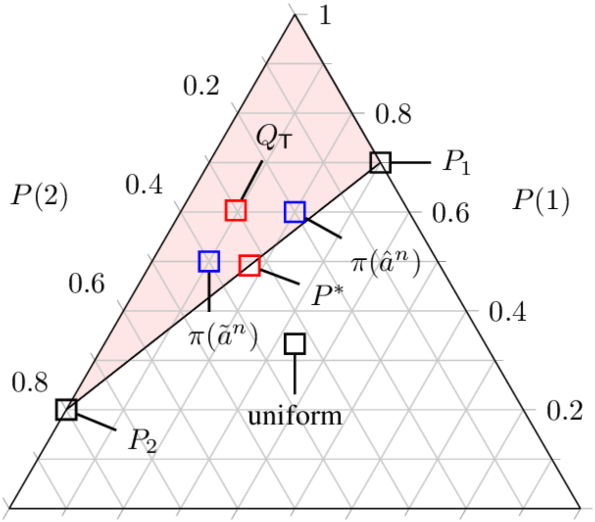
- Domain: $\Delta(A_{-i})$
- Range: subsets of $\Delta(A_i)$
- For finite games: nonempty, convex-valued, upper hemicontinuous
- These properties are key to existence ideas later

Upper Hemicontinuity: Intuition

If $\pi_{-i}^{k} \to \pi_{-i}$ and $\pi_i^k \in BR_i(\pi_{-i}^{k})$ with $\pi_i^k \to \pi_i$, then $\pi_i \in BR_i(\pi_{-i})$.

Small changes in beliefs do not create discontinuous jumps in optimal responses.

Visualizing BR: Probability Simplex



- Opponents' mixture on the horizontal axis
- Your BR as regions or line segments

Dominance and BR

- Strictly dominated actions are never best responses.
- Iterated elimination of strictly dominated strategies simplifies BR maps.
- Order independence holds for strict dominance.

Example 1: Coordination BRs

	L	R
U	(3,2)	(0,0)
D	(0,0)	(2,3)

- Mutual BR at (U,L) and at (D,R).
- Two pure NE arise.

Example 2: Prisoner's Dilemma BRs

	C	D
C	(3,3)	(0,5)
D	(5,0)	(1,1)

- D strictly dominates C for both.
- Unique mutual BR at (D,D).

Example 3: Chicken BRs

	Swerve	Straight
Swerve	(0,0)	(-1,1)
Straight	(1,-1)	(-M,-M)

- Two off-diagonal pure NE when M is large.
- No dominant strategies.

Example 4: Matching Pennies BRs

	H	T
H	(1,-1)	(-1,1)
T	(-1,1)	(1,-1)

- BRs cycle.
- No pure NE.
- Mixed NE required.

From BR to NE: Core Idea

A strategy profile π^* is a Nash equilibrium if for all i,

$$\pi_i^* \in BR_i(\pi_{-i}^*).$$

- No unilateral profitable deviation
- Mutual best response characterization

Nash Equilibrium: Formal Definition

In finite $(N, \{A_i\}, \{R_i\})$, π^* is a Nash equilibrium if for all i and all π_i,

$$R_i(\pi_i^*, \pi_{-i}^*) \geq R_i(\pi_i, \pi_{-i}^*).$$

- Pure NE if each π_i^* is a point mass
- Mixed NE if some π_i^* is a distribution

NE as Fixed Point of BR

Define $BR(\pi) = \times_i BR_i(\pi_{-i})$ across players.
Nash equilibria are fixed points of BR:

$$\pi^* \in BR(\pi^*).$$

Invariance and Normalization

- Positive affine transformations of a player's payoff preserve BR and NE.
- Normalize scales for convenience.
- Cross-player mixing of scales does not matter for equilibrium structure.

Mixed Strategies: Expected Utility

- Extend payoffs linearly in mixed strategies.
- For 2×2, let row play U with probability p and column play L with q.
- Compute expected payoffs for each pure action, then use indifference.

	L	R
U	(3,2)	(0,0)
D	(0,0)	(2,3)

Indifference Principle

At a mixed NE, each action in the support yields the same expected payoff.

- Equalize payoffs of supported actions.
- Out-of-support actions do not exceed that payoff.

Mixed NE: Matching Pennies

Row plays H with p, column plays H with q:

$$u_R(H) = 2q - 1, \quad u_R(T) = 1 - 2q \ \Rightarrow \ q = \frac{1}{2}.$$

By symmetry, $p = \frac{1}{2}$.

- Value: 0 for both players

Mixed NE: Battle of the Sexes

Payoffs:

- Ballet together: $(2, 1)$
- Football together: $(1, 2)$

Let row pick Ballet with p, column pick Ballet with q:

$$u_R(B) = 2q, \quad u_R(F) = 1(1-q) \;\Rightarrow\; q = \tfrac{1}{3},$$
$$u_C(B) = 1 \cdot p, \quad u_C(F) = 2(1-p) \;\Rightarrow\; p = \tfrac{2}{3}.$$

Mixed NE is $(p^*, q^*) = \left(\tfrac{2}{3}, \tfrac{1}{3}\right)$.

2x2 Mixed NE: Template

For

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix},$$

let the row player play U with probability p, and the column player play L with probability q.

- Row indifference: $aq + b(1-q) = cq + d(1-q) \;\Rightarrow\;$ solve for q
- Column indifference: $ep + g(1-p) = fp + h(1-p) \;\Rightarrow\;$ solve for p

Mutual BR Characterization

- In pure strategies: a cell is an NE if it is a best response for both players
- In mixed strategies: supports produce equal expected payoffs

Degeneracy and Tie Breaking

- Degenerate equilibria have multiple BR at the boundary
- Small perturbations can select a unique equilibrium
- Good practice for robustness checks

Equilibrium Selection

	Description	Example
Risk dominance	Largest “basin of attraction”. Risk of others’ mistakes $\rightarrow$ risk-dominant strategy	Hare in Stag Hunt: lower, but guaranteed payoff
Payoff dominance	Payoffs are as good as in other NEs, but someone is strictly better off	Stag in Stag Hunt: significantly higher reward, if successful

Zero-Sum Preview: Minimax

- In 2-player zero-sum, NE equals minimax solution.
- Value of the game is the row player's equilibrium payoff.
- Solvable by linear programming.
- Details in a later week.

Linear Programming Formulation (Row Player)

Let A be the row player's payoff matrix. Solve

$$\max_{x,v} \; v \quad \text{s.t.} \quad A^\top x \geq v\mathbf{1}, \;\; x \geq 0, \;\; \mathbf{1}^\top x = 1.$$

- The dual problem gives the column player's formulation
- The solution returns both the equilibrium strategy and the game value

Outline

Battle of the Sexes (2×2, mixed + 2 pure)

```
import numpy as np, nashpy as nash

A = np.array([[2,0],[0,1]])
B = np.array([[1,0],[0,2]])
G = nash.Game(A,B)

def show(eqs):
    for sa, sb in eqs:
        payA, payB = G[sa, sb]
        print(f"Row: {np.round(sa,3)}, Col: {np.round(sb,3)} -> "
              f"payoffs ({payA:.3f}, {payB:.3f})")
eqs = list(G.support_enumeration())
show(eqs)
# Mixed NE should be (p,q) = (2/3, 1/3)
```

- Enumerates supports and checks indifference
- Returns pure and mixed equilibria where they exist

Chicken / Hawk–Dove (2×2, mixed + 2 pure)

```python
import numpy as np, nashpy as nash

A = np.array([[ 0,-1],[ 1,-10]])
B = np.array([[ 0, 1],[-1,-10]])

G = nash.Game(A,B)
for sa, sb in G.support_enumeration():
    payA, payB = G[sa, sb]
    print(f"Row: {np.round(sa,3)}, Col: {np.round(sb,3)} -> \\
      ({payA:.3f}, {payB:.3f})")
# Expect two pure off-diagonal + one interior mixed.
```

Matching Pennies (2×2 zero-sum, unique mixed only)

```
import numpy as np, nashpy as nash

A = np.array([[ 1,-1],[-1, 1]])
G = nash.Game(A, -A)

for sa, sb in G.support_enumeration():
    payA, payB = G[sa, sb]
    print(f"(Row,Col) = ({np.round(sa,3)}, {np.round(sb,3)}) -> \\
      {payA:.3f}, {payB:.3f}")
# Unique mixed: ([0.5, 0.5], [0.5, 0.5])
```

Rock–Paper–Scissors (3×3 zero-sum, unique mixed only)

```
import numpy as np, nashpy as nash

A = np.array([[ 0,-1, 1],
              [ 1, 0,-1],
              [-1, 1, 0]])
G = nash.Game(A, -A)

for sa, sb in G.vertex_enumeration():
    payA, payB = G[sa, sb]
    print(f"Row: {np.round(sa,3)}, Col: {np.round(sb,3)} -> \\
     ({payA:.3f}, {payB:.3f})")
# Unique mixed: each player (1/3, 1/3, 1/3)
```

Non-uniform mixed in 3×3 zero-sum

```
import numpy as np, nashpy as nash

# Slightly biased RPS; equilibrium stays interior but not uniform.
A = np.array([[ 0, -1,  1.2],
              [ 1.0, 0, -1],
              [-1.2, 1,  0]])
G = nash.Game(A, -A)

for sa, sb in G.vertex_enumeration():
    payA, payB = G[sa, sb]
    print(f"Row: {np.round(sa,3)}, Col: {np.round(sb,3)} -> \\
      ({payA:.3f}, {payB:.3f})")
# Expect mixed with probabilities != 1/3
```

General-sum 3×3 with a support-size-2 mixed NE

```python
import numpy as np, nashpy as nash

A = np.array([[3, 0, 2],[0, 2, 3],[2, 3, 0]], dtype=float)
B = np.array([[2, 3, 0],[3, 0, 2],[0, 2, 3]], dtype=float)
G = nash.Game(A,B)

for sa, sb in G.support_enumeration():
    payA, payB = G[sa, sb]
    support_row = np.flatnonzero(sa > 1e-9)
    support_col = np.flatnonzero(sb > 1e-9)
    print(f"Row: {np.round(sa,3)} (supp {support_row}), "
          f"Col: {np.round(sb,3)} (supp {support_col}) "
          f"-> ({payA:.3f}, {payB:.3f})")
# Typically finds a mixed with support size 2 for each.
```

Validating Returned Equilibria

Given (π_1, π_2):

1. Compute $u_i = R_i(\pi_1, \pi_2)$.
2. Compute $u_i^{BR} = \max_{\pi_i'} R_i(\pi_i', \pi_{-i})$.
3. Check that $u_i^{BR} - u_i = 0$ up to tolerance.

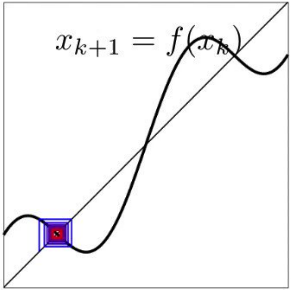
Outline

1. Banach Fixed Point Theorem (Contraction Mapping Theorem)

Statement:
Let (X, d) be a nonempty complete metric space and $f : X \to X$ a contraction (i.e., there exists $0 < c < 1$ such that $d(f(x), f(y)) \leq c \cdot d(x, y)$ for all x, y). Then there exists a unique fixed point x^* in X such that $f(x^*) = x^*$.

- Fundamental for proving convergence, but *not* generally used for Nash equilibrium existence, because game-theoretic best response correspondences are not contractions or even single-valued.

1. Banach Fixed Point Theorem



Attracting fixed point: $|f'(x^\star)| < 1.$

Repulsive fixed point: $|f'(x^\star)| > 1.$

Theorem: If $(\mathcal{X}, d)$ is a complete metric space, $f : \mathcal{X} \mapsto \mathcal{X}$ with $d(f(x), f(y)) \leqslant \kappa d(x, y)$ and $\kappa < 1$ then $\exists! x^\star$ such that $f(x^\star) = x^\star$.

$$x_{k+1} \stackrel{\text{def.}}{=} f(x_k) \xrightarrow{k \to +\infty} x^\star \qquad d(x_k, x^\star) \leqslant \kappa^k d(x_0, x^\star)$$

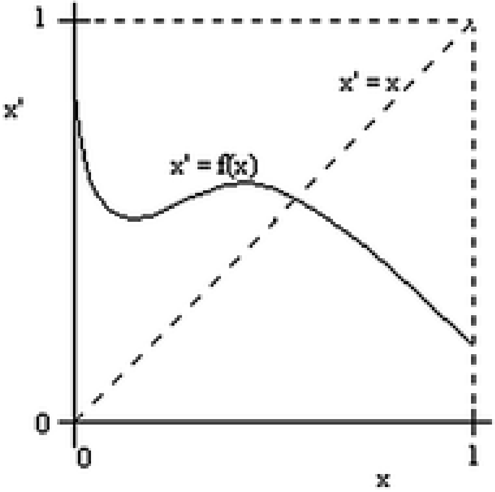
2. Brouwer Fixed Point Theorem

Statement:
Let $S \subseteq \mathbb{R}^n$ be nonempty, compact, and convex.
If $f : S \to S$ is continuous, then there exists $x^* \in S$ such that $f(x^*) = x^*$.

- Applies to single-valued, continuous functions.
- Nash's early proofs for continuous strategies (but not for general, set-valued best responses) used Brouwer.

Most best response functions in games are not continuous or are set-valued, so Brouwer does not directly apply in general.

2. Brouwer Fixed Point Theorem Plot



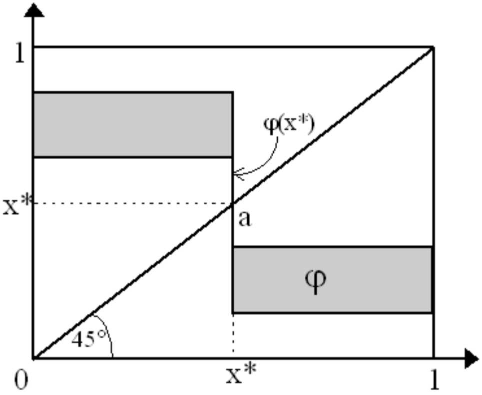
3. Kakutani Fixed Point Theorem

Statement:
Let $S \subseteq \mathbb{R}^n$ be nonempty, compact, and convex.
Let $\Phi : S \to 2^S$ be an upper hemicontinuous set-valued function (i.e., a correspondence) such that

1. For all $x \in S : \Phi(x)$ is nonempty, convex, and closed.
2. The graph of Φ is closed: If $x^k \to x$, $y^k \to y$, and $y^k \in \Phi(x^k)$, then $y \in \Phi(x)$.
   Then: There exists $x^* \in S$ such that $x^* \in \Phi(x^*)$.

- This is the key fixed point theorem for proving Nash equilibrium in games with mixed strategies, because the best response correspondence may be set-valued and upper hemicontinuous.

3. Kakutani Fixed Point Theorem Plot



Relationship & Hierarchy

- **Brouwer** is a special case of Kakutani (when the correspondence is single-valued, i.e., a function).
- **Banach** is fundamentally different, deals with iterative contractions. Not generally present in game theoretic contexts.
- In Nash's context, *Kakutani* is needed due to set-valuedness of the best response correspondence.

Nash Equilibrium Existence: Proof Sketch

Let $G = (N, (A_i), (R_i))$ be a finite normal-form game.

Strategy space:

- For each player i, the set of mixed strategies is the probability simplex over A_i, noted $\Delta(A_i)$.
- Joint strategy space is the product $S = \prod_i \Delta(A_i)$.
- S is compact and convex.

Best Response Correspondence:

- Define $\mathrm{BR}_i : S_{-i} \to 2^{S_i}$ by $\mathrm{BR}_i(\sigma_{-i}) = \arg\max_{\sigma_i \in S_i} R_i(\sigma_i, \sigma_{-i})$, where R_i is extended by linearity to mixed profiles.
- The best response correspondence for all players is $\mathrm{BR} : S \to 2^S$, mapping σ to the product set $\prod_i \mathrm{BR}_i(\sigma_{-i})$.

Nash Equilibrium Existence: Proof Sketch (Cont.)

Verification of Kakutani's Conditions:

- For each player, $\mathrm{BR}_i(\sigma_{-i})$ is nonempty (maximum of a continuous function over a compact simplex exists).
- Convexity: The set of maximizers of a linear function over a simplex is convex.
- Upper hemicontinuity (Closed Graph): Follows from Berge's Maximum Theorem because the payoff functions are continuous.
- Strategy space is a nonempty, compact, convex subset of $\mathbb{R}^k$.

Applying Kakutani:

- By Kakutani's Theorem, there exists $\sigma^* \in S$ with $\sigma^* \in \mathrm{BR}(\sigma^*)$.
- Therefore, every finite game has at least one mixed Nash equilibrium.

Why Is Proving Existence Hard?

- **Best response** maps are *set-valued* (correspondences), not single-valued functions.
- **Continuity issues:** Best response is not continuous as a function.
- **Compactness and convexity** of the strategy space is essential; non-convexity can lead to non-existence.
- **Upper hemicontinuity** and nonempty convex values are subtle properties, mathematically nontrivial to prove for arbitrary correspondences.
- Early attempts (pure strategies only) often fail, since no NE need exist in pure strategies.

Upper Hemicontinuity and Graph

- The graph of a set-valued function $\Phi : S \to 2^S$ is $\{(x, y) \mid x \in S, y \in \Phi(x)\}$.
- Φ is *upper hemicontinuous* at x if, whenever $x^k \to x$ and $y^k \in \Phi(x^k)$ with $y^k \to y$, it follows that $y \in \Phi(x)$.
- This ensures "no sudden jumps," which is crucial for fixed point existence.

Intuition: Why Mixed Strategies?

- The set of pure strategies is not convex or compact, so fixed-point theorems do not apply.
- Mixed strategies complete the set, ensuring compactness and convexity of the feasible region.

Table: Summary of Theorems

Theorem	Domain Type	Map Type	Existence	Uniqueness
Banach	Complete metric space	Contraction	Yes	Yes
Brouwer	Compact convex subset, $\mathbb{R}^n$	Continuous function	Yes	No
Kakutani	Compact convex subset, $\mathbb{R}^n$	Upper hemicontinuous correspondence	Yes	No

In what games is the best response function not single-valued? Give an example.

Why does convexity of the mixed strategy space matter for the application of Kakutani's theorem?

Can you give a simple set-valued function that violates upper hemicontinuity, and show what goes wrong?

Outline

Epsilon-Nash: Definition

A profile π is an ε-Nash equilibrium if for all i,

$$R_i(\pi) \;\geq\; \max_{\pi_i'} R_i(\pi_i', \pi_{-i}) - \varepsilon.$$

- $\varepsilon = 0$ gives an exact Nash equilibrium
- Useful when using numerical solvers or rounding

Why Epsilon-Nash Matters

- Rounding effects in computation
- Approximate rationality in practice
- Many algorithms converge to epsilon-NE rather than exact NE

Measuring Epsilon in Finite Games

For each player i:

1. Compute u_i at (π_1, π_2).
2. Compute the best pure-response payoff u_i^{BR}.
3. Set $\varepsilon_i = u_i^{BR} - u_i$.

Report $\max_i \varepsilon_i$.

Example: Epsilon for Rounded Matching Pennies

True NE: $(p, q) = (0.5, 0.5)$.
Use $(p, q) = (0.55, 0.45)$ for both players.

For the row player:

- If Row plays H:
  $u_R(H) = 0.45 \cdot 1 + 0.55 \cdot (-1) = -0.1.$
- If Row plays T:
  $u_R(T) = 0.55 \cdot 1 + 0.45 \cdot (-1) = 0.1.$
- If Row mixes with $p = 0.55$:
  $u_R = 0.55 \cdot (-0.1) + 0.45 \cdot (0.1) = -0.01.$

Example: Epsilon for Rounded Matching Pennies (Cont.)

Best-response payoff is 0.1 (by playing T).
Gap is $0.1 - (-0.01) = 0.11$.

By symmetry, the same holds for the column player.

NashPy: Compute Epsilon

```
import numpy as np

A = np.array([[1,-1],[-1,1]])
pi_row = np.array([0.55, 0.45])
pi_col = np.array([0.45, 0.55])

row_pure_payoffs = A @ pi_col
col_pure_payoffs = (-A).T @ pi_row

row_gap = row_pure_payoffs.max() \\
  - row_pure_payoffs @ pi_row
col_gap = col_pure_payoffs.max() \\
  - col_pure_payoffs @ pi_col
print("eps_row=", row_gap, "eps_col=", col_gap)
```

Parametric Games

- Vary a parameter in payoffs and track how the NE moves
- Useful for comparative statics
- Example: scale a penalty in Chicken and observe the mixed NE threshold

Parametric Example: Battle of the Sexes

Let Ballet payoffs be $(2, 1)$ and Football payoffs be $(\alpha, 2)$.

- Row indifference fixes $q(\alpha)$
- Column indifference fixes $p(\alpha)$
- Plot $p(\alpha)$ and $q(\alpha)$ to visualize shifts in mixing

Parametric Example: Solution

1. Row Indifference ($q(\alpha)$):

Set Row's expected payoff from Ballet and Football equal

$$2q = \alpha(1-q)$$

$$2q + \alpha q = \alpha$$

$$q(2+\alpha) = \alpha$$

$$q(\alpha) = \frac{\alpha}{2+\alpha}$$

2. Column Indifference ($p(\alpha)$):

Set Column's expected payoff from Ballet and Football equal

$$p = 2(1-p)$$

$$p + 2p = 2$$

$$3p = 2$$

$$p(\alpha) = \frac{2}{3}$$

Parametric Example: Solution (Cont.)

- Row's mixing probability: $q(\alpha) = \frac{\alpha}{2+\alpha}$
- Column's mixing probability: $p(\alpha) = \frac{2}{3}$

As α increases, $q(\alpha)$ increases from 0 toward 1.

$p(\alpha)$ remains fixed at $2/3$.

Practice 1

Identify all mutual BR cells in:

	L	R
U	(3,2)	(0,1)
D	(2,0)	(1,3)

1. Does a pure NE exist?
2. If not, solve for mixed NE.
3. Normalize payoffs and confirm NE unchanged.

Practice 2

1. Construct a 2×2 general-sum game with exactly one mixed Nash equilibrium.
2. For the zero-sum game with payoff matrix $\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$, compute the value and the equilibrium mixing.
3. In Chicken, replace $(-10, -10)$ with $(-M, -M)$. Find the threshold M that yields a mixed Nash equilibrium.

Practice 3

1. Use NashPy to compute all equilibria for three random 2×2 general-sum games.
2. For each equilibrium, round the probabilities to two decimals and compute ε.
3. Create a PNG of best-response (BR) lines and mark the intersection for one game.

True or False?

1. Every 2×2 game has a pure NE.
2. A strictly dominated action can be part of a mixed NE support.
3. In zero-sum games, NE equals minimax.
4. Mixed NE always make players indifferent across all actions.

Answers

1. False. Matching Pennies is a counterexample.
2. False. Dominated actions never belong to the support.
3. True. By minimax duality.
4. False. Only across supported actions.

Outline

Proof: Best Response Upper Hemicontinuity

- The strategy space (product of mixed strategy simplices) is compact and convex.
- Each player's payoff is continuous and linear in their own mixed strategy.
- For any fixed profile of opponents' strategies, the set of best responses is the set of maximizers of a continuous linear function over a simplex, which is a nonempty convex set.
- **Berge's Maximum Theorem:** If $f(x, y)$ is continuous and the constraint correspondence $C(x)$ is continuous (upper hemicontinuous, compact-valued), then the value function $M(x) = \max_{y \in C(x)} f(x, y)$ is continuous, and the maximizer correspondence is upper hemicontinuous and compact-valued.
- Therefore, the best response correspondence in finite games is upper hemicontinuous, convex-valued, and nonempty-valued, satisfying Kakutani's conditions for a fixed point.

Existence in Concave Games

If u_i is continuous in all arguments and concave in i's own strategy on a compact convex set, then a Nash equilibrium exists.

- The best response correspondence is upper hemicontinuous and convex-valued.
- Kakutani's theorem applies, so an equilibrium exists.

Potential Games: Existence and Structure

A game is a **potential game** if there exists a function Φ such that for any unilateral deviation by player i:

$$R_i(a_i', a_{-i}) - R_i(a_i, a_{-i}) = \Phi(a_i', a_{-i}) - \Phi(a_i, a_{-i})$$

- Every finite potential game has at least one pure strategy Nash equilibrium (since Φ attains a maximum).
- Best response dynamics converge to a pure NE.

Supermodular Games: Monotone Best Response

A game is **supermodular** if each player's payoff has increasing differences in their own strategy and others' strategies.

- The best response correspondence is monotone (increasing in others' strategies).
- **Tarski's Fixed Point Theorem:** Any monotone function on a complete lattice has a smallest and largest fixed point.
- Supermodular games have smallest and largest pure strategy Nash equilibria.

Outline

Support Enumeration

- For 2×2 games, enumerate all possible supports (sets of actions played with positive probability) for each player (size 1 or 2).
- For each support pair, solve the indifference equations (equalize expected payoffs for actions in support).
- Check feasibility (probabilities in $[0, 1]$) and that out-of-support actions do not yield higher payoffs.
- This is implemented in NashPy and other solvers.

Vertex Enumeration

- Nash equilibria correspond to vertices of best response polytopes (sets defined by best response inequalities).
- Enumerate candidate vertices and test equilibrium conditions.
- Efficient for small games and certain classes (e.g., zero-sum, symmetric).

Example: General-Sum 2×2

	L	R
U	(4,1)	(0,2)
D	(1,0)	(2,3)

Let $p = \Pr[U]$, $q = \Pr[L]$.

Row indifference:

$$4q + 0(1-q) = 1q + 2(1-q) \implies 4q = 2 - q \implies q = \frac{2}{5}$$

Column indifference:

$$1p + 0(1-p) = 2p + 3(1-p) \implies p = 3(1-p) \implies p = \frac{3}{4}$$

Compute expected payoffs for each action and verify that out-of-support actions do not yield higher payoffs.

Example: Mixed and Pure Check

- Check if any pure cell is a mutual best response (i.e., both payoffs are maximal in their row/column).
- If none, use $(p, q) = (\frac{3}{4}, \frac{2}{5})$ as above.
- Verify that all equilibrium conditions are satisfied.

Example: Zero-Sum 3×3 Rock Paper Scissors

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

- By symmetry, the unique NE is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ for both players.
- Value of the game is 0.
- Vertex enumeration recovers this efficiently.

Board Problem: Compute Mixed NE

	L	R
U	(3,4)	(0,1)
D	(1,0)	(2,3)

1. Write indifference equations for p, q.
2. Solve for p^*, q^*.
3. Verify out-of-support inequalities.
4. Compute expected payoffs for each action.

Extra Questions

1. In a 2x2 game, if both players put positive probability on both actions at an equilibrium, what must be true about the expected payoffs of their actions in support
2. Define $\varepsilon_i = u_i^{BR} - u_i$. If $\varepsilon_i = 0.08$ for each player, is the profile an ε-NE with $\varepsilon = 0.08$?

Extra Answers

1. They must be equal (indifference principle).
2. Yes

Summary

- Best response correspondences define Nash equilibrium.
- Every finite game has a mixed Nash equilibrium.
- In 2×2 games, solve for mixed NE using indifference and feasibility.
- ε-Nash equilibrium measures how close a profile is to equilibrium.
- NashPy helps compute and check equilibria in practice.