Game Theory

Lecture 3: Correlated Equilibrium, Welfare, and Learning

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Recap

Welfare & Efficiency

Correlated Equilibrium (CE)

Exercises

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Correlated Equilibrium (CE)

Exercises

Course textbooks

- Bonanno, G. (2024). *Game Theory (3rd ed.)*. University of California, Davis. Received from: GT Book
- Axelrod, R. (1984). *The Evolution of Cooperation*. Basic Books. Received from: Axelrod Article
- Nisan, N., Roughgarden, T., Tardos, É., & Vazirani, V. V. (2007). *Algorithmic Game Theory*. Cambridge University Press. Received from: AGT Book
- Myerson, R. B. (1991). Game Theory: Analysis of Conflict. Harvard University Press. Received from: GT Book 2
- F. Christianos et al., *Multi-Agent Reinforcement Learning: Foundations and Modern Approaches*, 2023. Received from: MARL Book.pdf
- Shoham, Y., & Leyton-Brown, K. (2008). Multiagent Systems: Algorithmic, Game-Theoretic, and Logical Foundations. Cambridge University Press Received from: MARL Book.pdf
- `nashpy` documentation (readthedocs) Link: NashPy Docs

Previously on Lecture 2

- Defined and computed Nash equilibrium using fixed point and best response theory.
- Proved existence of NE for finite games using Kakutani's theorem.
- Introduced ε-Nash equilibria for approximate computation.
- Used NashPy for real game equilibrium finding.
- Emphasized the role of best response and upper hemicontinuity.

Lecture Overview

- Welfare & efficiency
- Correlated Equilibrium (CE): definition, LP, examples

Outline

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Motivation: Why Mixed Strategies Matter

- Some games lack pure strategy NE (e.g. Matching Pennies).
- Mixed strategies guarantee an equilibrium in all finite games.
- Rational randomization: optimal unpredictability in competition.

Best Response as a Correspondence

- Domain: $\Delta(A_{-i})$
- Range: subsets of $\Delta(A_i)$
- For finite games: nonempty, convex-valued, upper hemicontinuous
- These properties are key to existence ideas later

Upper Hemicontinuity: Intuition

If $\pi^k_{-i} \to \pi_{-i}$ and $\pi^k_i \in BR_i(\pi^k_{-i})$ with $\pi^k_i \to \pi_i$, then $\pi_i \in BR_i(\pi_{-i})$.

Small changes in beliefs do not create discontinuous jumps in optimal responses.

NE as Fixed Point of BR

Define $BR(\pi) = \times_i BR_i(\pi_{-i})$ across players.
Nash equilibria are fixed points of BR:

$$\pi^* \in BR(\pi^*).$$

2x2 Mixed NE: Template

For

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix},$$

let the row player play U with probability p, and the column player play L with probability q.

- Row indifference: $aq + b(1-q) = cq + d(1-q) \;\Rightarrow\;$ solve for q
- Column indifference: $ep + g(1-p) = fp + h(1-p) \;\Rightarrow\;$ solve for p

Relationship & Hierarchy

- **Brouwer** is a special case of Kakutani (when the correspondence is single-valued, i.e., a function).
- **Banach** is fundamentally different, deals with iterative contractions-not generally present in game theoretic contexts.
- In Nash's context, *Kakutani* is needed due to set-valuedness of the best response correspondence.

Epsilon-Nash: Definition

A profile π is an ε-Nash equilibrium if for all i,

$$R_i(\pi) \;\geq\; \max_{\pi_i'} R_i(\pi_i', \pi_{-i}) - \varepsilon.$$

- $\varepsilon = 0$ gives an exact Nash equilibrium
- Useful when using numerical solvers or rounding

What changes today

- From **prediction** to **prescription**: efficiency and welfare
- From **independent** mixing to **correlated** signals (CE)
- From perfect rationality to **bounded** rationality (QRE)
- From static solutions to **learning dynamics**

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Formal setup: feasible payoffs

- Finite normal-form game with payoff functions $u_i(a)$ for $a \in A = \prod_i A_i$.
- Let X be the set of **joint distributions** on A (mixed/correlated play).
- **Feasible payoff set**:

$$U = \{(\mathbb{E}_x[u_1(a)], \dots, \mathbb{E}_x[u_n(a)]) : x \in X\}.$$

- U is **compact**; if mixed/correlated are allowed, U is the convex hull of the pure payoff vectors.

Pareto efficiency (weak / strong)

- $v, w \in U$.
- **Weak Pareto dominance**: $v \succeq_P w$ if $v_i \geq w_i$ for all i.
- **Strong dominance**: $v \succ_P w$ if $v \succeq_P w$ and $v_j > w_j$ for some j.
- **Pareto efficient** (PE): $w \in U$ is PE if there is no $v \in U$ with $v \succ_P w$.
- The set of PE points = the **upper-right boundary** (outer frontier) of U.

Existence & geometry of the frontier

- In finite games with correlated play allowed, $U = \mathrm{conv}\{u(a) : a \in A\}$ is a **polytope**.
- The **Pareto frontier** is nonempty and **closed** (upper boundary of a compact set).
- **Extreme efficient points** arise from optimizing linear aggregates $\sum_i \lambda_i u_i$, $\lambda \geq 0$.

Computing the frontier: weighted sums

For weights $\lambda \in \mathbb{R}^n_{\geq 0}$, solve

$$\max_{x \in X} \sum_{a \in A} x(a) \left(\sum_i \lambda_i u_i(a) \right) \quad \text{s.t.} \quad \sum_a x(a) = 1, \; x(a) \geq 0.$$

- Vary λ to **trace** (outer) frontier.
- If you **restrict to product mixes** (x independent), you still get a convex set in payoffs; allowing full correlation can **expand** U.

Scalarization completeness (convex case)

- If U is convex, then **every** PE point can be obtained by some **weighted-sum** scalarization with nonnegative weights.
- If one restricts to **pure** or **product** strategies only, U may be nonconvex; some PE points then require ε**-constraint** or explicit convexification.

Social welfare objectives: three classics

- **Utilitarian**: maximize total surplus $\sum_i u_i$.
- **Egalitarian**: maximize $\min_i u_i$.
- **Nash social welfare (NSW)**: maximize $\prod_i (u_i - \bar{u}_i)$ for some baseline $\bar{u}$ (e.g., disagreement); equivalent to $\max \sum_i \log(u_i - \bar{u}_i)$ when positive.

Utilitarian program (mixed/correlated)

$$\max_{x \in X} \sum_a x(a) \sum_i u_i(a) \quad \text{s.t.} \quad \sum_a x(a) = 1,\ x(a) \geq 0.$$

- Linear program over the **correlated** simplex.
- Returns a point on the frontier that maximizes **total** welfare.

Egalitarian (max–min) via epigraph trick

$$\max_{t,\, x \in X} t \quad \text{s.t.} \quad \sum_a x(a) u_i(a) \;\geq\; t \quad \forall i, \; \sum_a x(a) = 1, \; x(a) \geq 0.$$

- Linear program (if utilities are linear in x).
- Produces a **balanced** PE point.

Nash social welfare (convex form)

Given baselines $\bar{u}_i$ with feasibility $u_i > \bar{u}_i$, solve

$$\max_{x \in X} \sum_{i=1}^{n} \log\Big(\sum_a x(a)u_i(a) - \bar{u}_i\Big)$$

- Concave in x (sum of concave $\log$ of affine functions).
- Yields the **Nash bargaining** point under standard axioms.

Example: Pareto sets in 2×2 (BoS)

	Ballet	Football
Ballet	(2,1)	(0,0)
Football	(0,0)	(1,2)

- Feasible payoffs (with correlation) lie in $\mathrm{conv}\{(2,1),(1,2),(0,0)\}$.
- **Efficient** boundary $=$ the segment between $(2,1)$ and $(1,2)$.
- NE: two pure extreme points plus the mixed interior (inefficient vs risk).
- CE can reach any point on that segment (e.g., fair $(1.5,1.5)$).

Example: PD - stability vs efficiency

	C	D
C	(3,3)	(0,5)
D	(5,0)	(1,1)

- (D, D) is the **unique NE**, but **Pareto-dominated** by (C, C).
- Frontier includes $(3, 3)$ and the upper-right part of $\text{conv}\{(3, 3), (5, 0), (0, 5)\}$.
- **Proof of inefficiency of NE**: Since D strictly dominates C for both, the unique NE is (D, D).

Example: Stag Hunt - payoff vs risk dominance

	Stag	Hare
Stag	(3,3)	(0,2)
Hare	(2,0)	(2,2)

- **Efficient** point: $(3, 3)$ (also PE frontier's top corner).
- NE: $(\text{Stag}, \text{Stag})$ and $(\text{Hare}, \text{Hare})$.
- **Risk-dominance**: $(\text{Hare}, \text{Hare})$ has larger basin under noise; but it is **inefficient** vs $(\text{Stag}, \text{Stag})$.

Example: Chicken - avoiding catastrophe

	Swerve	Straight
Swerve	(0,0)	(-1,1)
Straight	(1,-1)	(-M,-M)

- For $M \gg 1$, the lower-right is catastrophic; PE frontier lies along off-diagonal payoffs.
- Any equilibrium **placing mass** on $(-M, -M)$ is **highly inefficient**; CE can put **zero** mass there and remain incentive compatible.

Zero-sum games & efficiency

- If $u_1 = -u_2$, then for any feasible (u_1, u_2), $u_1 + u_2 = 0$ (sum welfare is **constant**).
- The **Pareto frontier** is the anti-diagonal; **every** feasible point is weakly PE (improving one hurts the other).
- Hence "efficiency" is trivial under utilitarian welfare; the **value** (minimax) is the relevant benchmark.

Price of Anarchy (PoA): general

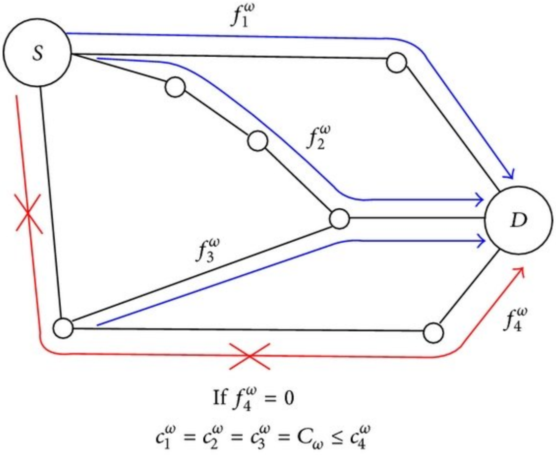
Let $W(s)$ be welfare (higher is better). For a game with strategy space S:

$$\text{PoA} = \frac{\max_{s \in S} W(s)}{\min_{s \in \text{NE}} W(s)} \in (0, 1].$$

- If cost C is minimized, use $\text{PoA} = \dfrac{\max_{s \in \text{NE}} C(s)}{\min_{s \in S} C(s)} \geq 1$.
- Quantifies **worst-case efficiency loss** due to strategic behavior.

Nonatomic congestion (Wardrop) model

- Continuum of infinitesimal users of total demand 1.
- Parallel links e with latency $\ell_e(x_e)$ depending on flow x_e.
- **Wardrop equilibrium**: all used routes have equal (minimal) latency.
- **Social optimum**: minimize total latency $\sum_e x_e \ell_e(x_e)$.



Efficiency under correlation

- Allowing **correlated** signals can **move** play toward the Pareto frontier in general-sum games.
- In **zero-sum**, correlation doesn't improve total welfare (value fixed).
- In **congestion** settings, mechanism tweaks (tolls, signals) can **implement** efficient flows.

KKT view of the frontier

- Frontier points solve

$$\max_{x \in X} \sum_i \lambda_i u_i(x) \quad \text{with} \quad \lambda \in \mathbb{R}^n_{\geq 0}.$$

- KKT multipliers λ act as **social prices** on individual utilities.
- The **supporting hyperplane** with normal λ touches U at PE points.

When NE is PE (and when not)

- NE can be PE (e.g., common-interest games with unique maximizer).

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Aumann's idea: recommendations you want to obey

- A **mediator** draws a joint action $a = (i, j)$ from a public distribution x on $A_1 \times A_2$, and sends **private** recommendation i to Row and j to Column.
- Each player updates by Bayes:

$$\Pr(j \mid i) = \frac{x_{ij}}{\sum_{k \in A_2} x_{ik}}.$$

- A **Correlated Equilibrium (CE)** is any x such that *obeying the recommendation* is a best response given the **posterior** they infer from their own signal.

CE obedience constraints

Let $u_1 = A$, $u_2 = B$. For Row, after receiving i, the expected gain from obeying i vs deviating to i' is

$$\sum_j \Pr(j \mid i)\,(A_{ij} - A_{i'j}).$$

Multiply by $\Pr(i) = \sum_j x_{ij}$ to avoid division by zero:

$$\sum_j x_{ij}\,(A_{ij} - A_{i'j}) \;\geq\; 0 \quad \forall i, i'.$$

Similarly for Column:

$$\sum_i x_{ij}\,(B_{ij} - B_{ij'}) \;\geq\; 0 \quad \forall j, j'.$$

Plus $x_{ij} \geq 0$ and $\sum_{ij} x_{ij} = 1$.

Sets & inclusions

- Let **NE** be mixed Nash distributions (independent product mixes over actions).
- Let **CE** be all obeyable joint distributions x (private signals allowed).
- Let **CCE** (coarse correlated equilibrium) relax to one-shot deviations **before** seeing the signal:
  - Row: $\sum_{ij} x_{ij} (A_{ij} - A_{i'j}) \geq 0 \; \forall i'$.
  - Column: $\sum_{ij} x_{ij} (B_{ij} - B_{ij'}) \geq 0 \; \forall j'$.
- Then:

$$\text{NE} \subseteq \text{CE} \subseteq \text{CCE}.$$

- **Why strict?** CE can correlate actions to avoid miscoordination; CCE is even larger since players can't condition on the signal when deviating.

Geometry: the CE polytope

- Variables $x \in \mathbb{R}^{mn}$ with simplex constraints $x \geq 0$, $\mathbf{1}^\top x = 1$.
- Add **linear** obedience inequalities $\to$ a **convex polytope** X_{CE}.
- **Extreme points** of X_{CE} need not be product distributions; can be "purely correlated".
- For a 2-player $m \times n$ game, any extreme CE has support size $\leq m + n - 1$ (Carathéodory/linear-independence argument).

NE are CE: quick proof

If $x = p \otimes q$ is a **mixed NE**, each action in support is a best response to the **independent** posterior. Then for any deviation i',

$$\sum_j x_{ij}(A_{ij} - A_{i'j}) = p_i \sum_j q_j(A_{ij} - A_{i'j}) \geq 0,$$

since i is a best response to q. Same for Column. Hence $x \in$ CE.

Zero-sum invariance

If $B = -A$ (two-player zero-sum), all CE yield Row payoff $\leq v$ and $\geq v$, where v is the minimax value. So **every CE attains value v**:

- Row's CCE constraint implies $\sum_{ij} x_{ij} A_{ij} \geq \max_{i'} \sum_j x_{\cdot j} A_{i'j}$.
- Column's implies $\sum_{ij} x_{ij} A_{ij} \leq \min_{j'} \sum_i x_{i\cdot} A_{ij'}$.
- Sandwiching between minmax and maxmin gives equality at v. **Conclusion:** CE cannot help in strictly competitive games; it can help a lot in coordination.

CE via LP

Decision vars: $x_{ij} \geq 0$, $\sum_{ij} x_{ij} = 1$.
Incentive constraints: as above.
Objective: choose your scalarization:

- **Utilitarian**: $\max \sum_{ij} x_{ij}(A_{ij} + B_{ij})$.
- **Egalitarian**: epigraph trick for $\max \min\{\sum_{ij} x_{ij} A_{ij}, \sum_{ij} x_{ij} B_{ij}\}$.
- **Fairness**: $\max \sum_{ij} x_{ij}(\alpha A_{ij} + \beta B_{ij})$ with $\alpha = \beta$.

Solvable in polynomial time; returns a CE distribution and induced payoffs.

CE: Battle of the Sexes

	Ballet	Football
Ballet	(2,1)	(0,0)
Football	(0,0)	(1,2)

Take $x_{BB} = x_{FF} = 1/2$, others 0.

- **Row obedience:** comparing B vs F:
  $\sum_j x_{Bj}(A_{Bj} - A_{Fj}) = x_{BB}(2-0) + x_{BF}(0-1) = 1.$
- **Column obedience:** comparing B vs F:
  $\sum_i x_{iB}(B_{iB} - B_{iF}) = x_{BB}(1-0) + x_{FB}(0-2) = 1.$

All other deviation pairs are slack/identical. Thus CE holds, payoffs $(1.5, 1.5)$.
Insight: CE **removes miscoordination risk**, unlike the mixed NE.

CE: Chicken with catastrophe M

	Swerve	Straight
Swerve	(0,0)	(-1,1)
Straight	(1,-1)	(-M,-M)

Let $x_{SS} = x_{TT} = 0$, $x_{ST} = x_{TS} = \frac{1}{2}$.

- Row: receiving S, Column posterior is T with prob 1 $\rightarrow$ obeying S yields 0 vs deviating to T yields $-M$ $\rightarrow$ obey.
- Row: receiving T, posterior is S with prob 1 $\rightarrow$ obeying T yields 1 vs deviating to S yields -1 $\rightarrow$ obey.

  Symmetric for Column. This CE **eliminates** $(-M, -M)$ entirely and gives welfare 0. For large M, this **strictly** dominates any mixed NE with crash probability.

PD: CE doesn't magically give (C,C)

	C	D
C	(3,3)	(0,5)
D	(5,0)	(1,1)

- Any CE must satisfy obedience: if Row recommended C, deviating to D against Column's posterior must not help.
- With mass on (C, C) and (D, D) only, Row's deviation from C to D against the posterior (prob 1 on C) **gains** $5 - 3 > 0 \rightarrow$ violates obedience.
- Thus (C, C) cannot be sustained by CE **without transfers or repeated-game incentives**. (CE helps in **coordination**, not in **dominant-strategy temptations** like PD.)

Hand-check template (2×2)

Suppose x has support on cells $\{(i_1, j_1), (i_2, j_2)\}$.

Row obedience for each received $i \in \{i_1, i_2\}$ vs deviating to i':

$$\sum_j x_{ij}(A_{ij} - A_{i'j}) \geq 0.$$

With two supported j's, this is two inequalities per i'.

1) **Column obedience** symmetric.
2) **Normalization** and **nonnegativity**.
3) Compute expected payoffs; compare to NE and to PE frontier.

CE vs CCE on risk and welfare

- **NE** mixes can put mass on **miscoordination** with bad outcomes (e.g., Chicken crash).
- **CE** can **correlate** to avoid jointly bad states while keeping incentives.
- **CCE** is larger; many **no-regret dynamics** converge to CCE, yielding **robust PoA** guarantees via smoothness.

Designing CE in the wild

- **Private prompts**: "If your friend chooses Ballet, we'll recommend Ballet to you," shown as a *personalized nudge*.
- **Public tie-breakers**: randomized "coin flips" everyone trusts, then **private** route recommendations.
- CE requires **credibility** that the mediator draws from the announced x and that messages are **private**.

CE and welfare: frontier positioning

- CE feasibility region is **convex**, often **strictly containing** NE.
- By maximizing a welfare functional over the CE polytope, you can **hit the Pareto frontier** in many coordination games (e.g., BoS midpoint).
- In zero-sum: no improvement in value; in general-sum: CE can **strictly improve** fairness.

Micro-LP (symbolic 2×2)

Let payoffs be

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}, \quad x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}.$$

Row obedience:

$$x_{11}(a-c) + x_{12}(b-d) \geq 0 \quad \text{(obey 1 vs dev 2 when rec 1)},$$
$$x_{21}(c-a) + x_{22}(d-b) \geq 0 \quad \text{(obey 2 vs dev 1 when rec 2)}.$$

Column obedience:

$$x_{11}(e-g) + x_{21}(g-e) \geq 0 \quad \text{(obey 1 vs dev 2 when rec 1)},$$
$$x_{12}(f-h) + x_{22}(h-f) \geq 0 \quad \text{(obey 2 vs dev 1 when rec 2)}.$$

Plus $x_{ij} \geq 0$, $\sum_{ij} x_{ij} = 1$.
Pick objective (e.g., maximize $\sum_{ij} x_{ij}(A_{ij} + B_{ij})$) and solve.

CE vs public correlation

- **Private** recommendations are sufficient for CE.
- With a **public** signal only (no private advice), you generally get a **public correlated equilibrium**; this can be weaker (players can infer others' advice and may want to deviate).
- Private messages are key to **obedience** at the individual level.

When CE fails to help

- **Dominance-driven** temptations (PD): obedience to "cooperate" is not credible.
- **Strictly competitive** (zero-sum): value fixed.
- **Miscoordinated posteriors**: your candidate x induces posteriors that make deviation profitable $\rightarrow$ not a CE.

CE & learning

- Many **no-regret** learning processes converge to the **CCE** set; with smoothness, their **worst-case welfare** matches PoA bounds.
- Adding **signal devices** (recommendations) can move play from CCE toward **CE** and closer to the **Pareto frontier**.

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Correlated Equilibrium in BoS (coin on diagonal)

Verify and analyze a CE that fairly coordinates play.

Given. Battle of the Sexes payoffs:

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Proposed joint distribution x: $x_{BB} = x_{FF} = \frac{1}{2}$, others 0.

1. Write the CE inequalities explicitly for row (B vs F, F vs B) and column (B vs F, F vs B).
2. Plug the proposed x into all CE inequalities and show they hold with slack (or equality).
3. Compute $\mathbb{E}[u_R], \mathbb{E}[u_C]$ under this CE.
4. Compare to (i) the two pure NE, and (ii) the mixed NE payoffs.

You should get $(\mathbb{E}u_R, \mathbb{E}u_C) = (1.5, 1.5)$. Mixed NE payoffs equal the same welfare but with miscoordination risk.

Summary

- Efficiency lenses: Pareto frontier, Social Welfare, Price of Anarchy
- Beyond NE: **CE/CCE** (obedience via signals)